On a conjecture of Ōshima

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Abstract

The set of homotopy classes of self maps of a compact, connected Lie group G is a group by the pointwise multiplication which we denote by $\mathcal{H}(G)$, and it is known to be nilpotent. Ōshima [9] conjectured if G is simple, then $\mathcal{H}(G)$ is nilpotent of class $\geq \mathrm{rank}G$. We show this is true for $\mathrm{PU}(p)$ which is the first high rank example.

1 Introduction and statement of the result

We will denote the class of a nilpotent group K by $\mathrm{nil}K$ and normalize it so that K is abelian if and only if $\mathrm{nil}K = 1$.

For based spaces X, Y, let $[X, Y]$ denote the set of based homotopy classes of based maps from X to Y. When Y is group-like, $[X, Y]$ has the natural group structure given by the pointwise multiplication. It is classical that if Y is connected and $\mathrm{cat}X < \infty$, then the group $[X, Y]$ is nilpotent of class $\leq \mathrm{cat}X$ [10], where $\mathrm{cat}X$ stands for the Lusternik-Schnirelmann category of X normalized as $\mathrm{cat}(*) = 0$.

For a group-like space X, we denote the group $[X, X]$ by $\mathcal{H}(X)$ and call it the self homotopy group of X. Let G be a compact, connected Lie group. Then, as noted above, the group $\mathcal{H}(G)$ is nilpotent of class $\leq \mathrm{cat}G$ and thus we have an invariant $\mathrm{nil}\mathcal{H}(G)$ for $\mathcal{H}(G)$. Ōshima and the second author [7] showed that, for most of compact, connected Lie groups G, $\mathcal{H}(G)$ is not abelian, that is, $\mathrm{nil}\mathcal{H}(G) \geq 2$. Then we address here the problem how far from being abelian $\mathcal{H}(G)$ is, that is, how big $\mathrm{nil}\mathcal{H}(G)$ is. In [9], Ōshima conjectured:

Conjecture 1. If G is a compact, connected, simple Lie group, then $\mathrm{nil}\mathcal{H}(G) \geq \mathrm{rank}G$.

This conjecture is false if we do not assume G is simple [9]. In some cases of rank ≤ 3, the above conjecture is known to be true (see [1]). However, if the rank of G is greater than 3, there have not been any example of G making this conjecture true. In fact, as is shown in [4] the projective unitary group $\mathrm{PU}(n)$ is the only one example of G having $\mathrm{nil}\mathcal{H}(G) \geq 6$ so far. More precisely, it is shown in [4] that

$$\mathrm{nil}\mathcal{H}(\mathrm{PU}(p)) \geq p - 2 = \mathrm{rank}\mathrm{PU}(p) - 1$$

for any odd prime p. The aim of this note is to improve this inequality by one to satisfy Ōshima's conjecture as:

Theorem 1.1. *For any prime p,* $\mathrm{nil}\mathcal{H}(\mathrm{PU}(p)) \geq \mathrm{rank}\mathrm{PU}(p)$.

2 Proof of Theorem 1.1

When $p = 2$, Theorem 1.1 is trivial and then we will assume the prime p is odd. We will implicitly use the naturality

$$[X, Y]_{(p)} \cong [X, Y_{(p)}] \cong [X_{(p)}, Y_{(p)}]$$

for a finite dimensional suspension X, where $-_{(p)}$ denotes the p-localization in the sense of Bousfield and Kan [3]. We will identify continuous maps with their homotopy classes. Since $\mathrm{PU}(p) \cong \mathrm{PSU}(p)$, we will also identify $\mathrm{PU}(p)$ with $\mathrm{PSU}(p)$.

We first collect facts on $\mathrm{SU}(p)$ which we will use. Let ϵ_k denote a generator of $\pi_{2k-1}(\mathrm{SU}(p)) \cong \mathbf{Z}$ for $2 \leq k \leq p$. Define a map $\mu : \prod_{k=2}^{p} S^{2k-1} \to \mathrm{SU}(p)$ by $\mu(x_2, \ldots, x_p) = \epsilon_2(x_2) \cdots \epsilon_p(x_p)$ for $(x_2, \ldots, x_p) \in \prod_{k=2}^{p} S^{2k-1}$. Then the classical result of Serre [11] shows that we have a homotopy equivalence:

$$\mu_{(p)} : \prod_{k=2}^{p} S^{2k-1}_{(p)} \xrightarrow{\simeq} \mathrm{SU}(p)_{(p)}$$

We will denote the composition of $\mu_{(p)}^{-1}$ and the i-th projection $\prod_{k=2}^{p} S^{2k-1}_{(p)} \to S^{2i-1}_{(p)}$ by λ_i. It is shown by Bott [2] that the order of the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is divisible by $\frac{(i+j-1)!}{(i-1)!(j-1)!}$. In particular, $\langle \epsilon_p, \epsilon_i \rangle_{(p)}$ is nontrivial for $2 \leq i \leq p$. Recall that we have, for $i \geq 2$,

$$\pi_{2i-1+k}(S^{2i-1}_{(p)}) \cong \begin{cases} \mathbf{Z}/p & k = 2p-3 \\ 0 & 0 < k < 4p-6 \text{ and } k \neq 2p-3 \end{cases} \tag{2.1}$$

in which $\pi_{2i+2p-4}(S^{2i-1}_{(p)})$ is generated by $\Sigma^{2i-4}\alpha_1$ for a generator α_1 of $\pi_{2p}(S^3_{(p)})$. Then it follows that, for $2 \leq i \leq p$,

$$\lambda_{2i+1} \circ \langle \epsilon_p, \epsilon_i \rangle_{(p)} \neq 0. \tag{2.2}$$

Now we construct a map from a lens space to $\mathrm{PU}(p)$. Let L be the lens space $S^{2p-1}/(\mathbf{Z}/p)$ and let $\pi : \mathrm{SU}(p) \to \mathrm{PU}(p)$ and $\rho : S^{2p-1} \to L$ be the projections.

Proposition 2.1. *There is a map $\epsilon : L_{(p)} \to \mathrm{PU}(p)_{(p)}$ satisfying the homotopy commutative diagram:*

$$\begin{tikzcd} S^{2p-1}_{(p)} \arrow[r, "\epsilon_{p(p)}"] \arrow[d, "\rho_{(p)}"'] & \mathrm{SU}(p)_{(p)} \arrow[d, "\pi_{(p)}"] \\ L_{(p)} \arrow[r, "\epsilon"] & \mathrm{PU}(p)_{(p)} \end{tikzcd}$$

Proof. We denote the projections $\mathrm{SU}(p) \to \mathrm{SU}(p)/\mathrm{SU}(p-1) = S^{2p-1}$ and $\mathrm{PU}(p) \to \mathrm{PU}(p)/\mathrm{SU}(p-1) = L$ by κ and $\bar{\kappa}$ respectively. Then we have $\bar{\kappa} \circ \pi = \rho \circ \kappa$. Recall that the cohomology of $\mathrm{SU}(p)$ and $\mathrm{PU}(p)$ are given by

$$H^*(\mathrm{SU}(p)) = \Lambda(x_3, x_5, \dots, x_{2p-1}),\ |x_j| = j.$$

and

$$H^*(\mathrm{PU}(p)) = \mathbf{Z}/p[y_2]/(y_2^p) \otimes \Lambda(y_1, y_3, \dots, y_{2p-3}),\ |y_j| = j$$

so that $\pi^*(y_{2i-1}) = x_{2i-1}$ for $2 \le i \le p-1$. Consider maps

$$\theta = \kappa_{(p)} \times \prod_{k=2}^{p-1} x_{2k-1} : \mathrm{SU}(p)_{(p)} \to S^{2p-1}_{(p)} \times \prod_{k=2}^{p-1} K(\mathbf{Z}_{(p)}, 2k-1)$$

and

$$\bar{\theta} = \bar{\kappa}_{(p)} \times \prod_{k=2}^{p-1} y_{2k-1} : \mathrm{PU}(p)_{(p)} \to L_{(p)} \times \prod_{k=2}^{p-1} K(\mathbf{Z}_{(p)}, 2k-1).$$

Then we have $(\rho \times 1)_{(p)} \circ \theta = \bar{\theta} \circ \pi_{(p)}$ and thus since θ is a $2p$-equivalence and $\bar{\kappa}_* : \pi_1(\mathrm{PU}(p)) \to \pi_1(L)$ is an isomorphism, $\bar{\theta}$ is a $2p$-equivalence. Note that L is of dimension $2p-1$. Then by the Whitehead theorem there is a map $\epsilon : L_{(p)} \to \mathrm{PU}(p)_{(p)}$, unique up to homotopy, so that $\bar{\kappa}_{(p)} \circ \epsilon = 1_{L_{(p)}}$. Thus we have $\bar{\theta} \circ \epsilon \circ \rho_{(p)} = \bar{\theta} \circ \pi_{(p)} \circ \epsilon_{p(p)}$ which implies $\epsilon \circ \rho_{(p)} = \pi_{(p)} \circ \epsilon_{(p)}$, and therefore we have established the proposition. □

Remark 2.1. It should be mentioned here that Hamanaka and the authors [4] have obtained the above map ϵ by decomposing $\mathrm{PU}(p)_{(p)}$. Harper [5] also constructed a map $L_{(p)} \to \mathrm{PU}(p)$ and one can verify that Harper's map satisfies the above homotopy commutative diagram by examining the homotopy groups. Both of the above works are generalized in [6].

Note that there is a map $\hat{\gamma} : \mathrm{PU}(p) \wedge \mathrm{SU}(p) \to \mathrm{SU}(p)$ such that $\hat{\gamma} \circ (\pi \wedge 1) = \gamma$ for the reduced commutator map $\gamma : \mathrm{SU}(p) \wedge \mathrm{SU}(p) \to \mathrm{SU}(p)$. Let L_k be the Moore space $S^{2k-1} \cup_p e^{2k}$ for $1 \le k \le p-1$ and S^{2p-1} for $k = p$. Then in particular L_1 is the 2-skeleton of L.

Lemma 2.1. *Let $q_k : L_k \to S^{2k}$ be the pinch map. Then, for $2 \le i \le p-1$, we have*

$$\lambda_{i+1} \circ \hat{\gamma}_{(p)} \circ (\epsilon|_{L_1} \wedge \epsilon_i)_{(p)} = a_i (q_1 \wedge 1_{S^{2i-1}})_{(p)},\ a_i \in \mathbf{Z}_{(p)}^\times.$$

Proof. Recall from [8] there is a homotopy equivalence

$$\Sigma L_{(p)} \simeq \vee_{k=1}^p \Sigma L_{k(p)}.$$

Then there are maps $f_k : S^{2p+2i-2}_{(p)} \to (L_k \wedge S^{2i-1})_{(p)}$ for $1 \le k \le p$ such that

$$(\rho \wedge 1_{S^{2i-1}})_{(p)} = \vee_{k=2}^p f_k. \tag{2.3}$$

Since ρ is a p-fold covering, we have $f_p = p$.

Consider the exact sequence

$$\begin{aligned}\pi_{2i+2k-1}(S^{2i+1}) &\xrightarrow{\times p} \pi_{2i+2k-1}(S^{2i+1}) \\ &\xrightarrow{q_k^*} [L_k \wedge S^{2i-1}, S^{2i+1}] \to \pi_{2i+2k-2}(S^{2i+1}) \xrightarrow{\times p} \pi_{2i+2k-2}(S^{2i+1})\end{aligned}$$

induced from the cofibre sequence $S^{2k-1} \xrightarrow{p} S^{2k-1} \to L_k \xrightarrow{q_k} S^{2k} \xrightarrow{p} S^{2k}$. Then by (2.1) we have:

$$[L_k \wedge S^{2i-1}, S^{2i+1}]_{(p)} \cong \begin{cases} \mathbf{Z}_{(p)} & k = 1 \\ 0 & 2 \le k \le p-1 \\ \mathbf{Z}/p & k = p \end{cases}$$

in which $[L_1 \wedge S^{2i-1}, S^{2i+1}]_{(p)}$ is generated by $(q_1 \wedge 1_{S^{2i-1}})_{(p)}$. Hence it follows that

$$\lambda_{i+1} \circ \hat{\gamma}_{(p)} \circ (\epsilon \wedge \epsilon_{i(p)}) = a_i (q_1 \wedge 1_{S^{2i-1}})_{(p)} \vee a_i' \Sigma^{2i-4} \alpha_1$$

for $a_i, a_i' \in \mathbf{Z}_{(p)}$ and $2 \le i \le p-1$. Thus by (2.2), Proposition 2.1 and (2.3) we obtain

$$\begin{aligned} 0 \neq \lambda_{i+1} \circ \langle \epsilon_p, \epsilon_i \rangle_{(p)} &= \lambda_{i+1} \circ \hat{\gamma}_{(p)} \circ (\epsilon \wedge \epsilon_{i(p)}) \circ (\rho \wedge 1_{S^{2i-1}})_{(p)} \\ &= a_i (q_1 \wedge 1_{S^{2i-1}})_{(p)} \circ f_1 \vee p a_i' \Sigma^{2i-4} \alpha_1 \\ &= a_i (q_1 \wedge 1_{S^{2i-1}})_{(p)} \circ f_1. \end{aligned}$$

It follows from (2.1) that $(q_1 \wedge 1_{S^{2i-1}})_{(p)} \circ f_1 = a \Sigma^{2i-4} \alpha_1$ for $a \in \mathbf{Z}/p$ and thus $a_i \in \mathbf{Z}_{(p)}^\times$. Therefore the proof is completed. □

We will use the same notation for the cohomology of $\mathrm{SU}(p)$ and $\mathrm{PU}(p)$ as in Proposition 2.1. Then by Lemma 2.1 and the Whitehead theorem we obtain:

Corollary 2.1. *Let I be the ideal $\bar{H}^*(\mathrm{PU}(p))^2 \otimes \bar{H}^*(\mathrm{SU}(p)) + \bar{H}^*(\mathrm{PU}(p)) \otimes \bar{H}^*(\mathrm{SU}(p))^2$ in $H^*(\mathrm{PU}(p) \wedge \mathrm{SU}(p))$. Then we have*

$$\hat{\gamma}^*(x_{2i+1}) \equiv b_i y_2 \otimes x_{2i-1} \mod I$$

for $b_i \in (\mathbf{Z}/p)^\times$.

Proof of Theorem 1.1. Put $\hat{\gamma}_{p-2} = \hat{\gamma} \circ (1 \wedge \hat{\gamma}) \circ \cdots \circ (\underbrace{1 \wedge \cdots \wedge 1}_{p-3} \wedge \hat{\gamma})$. It follows from Corollary 2.1 that

$$\hat{\gamma}_{p-2}^*(x_{2p-1}) = \underbrace{y_2 \otimes \cdots \otimes y_2}_{p-2} \otimes x_3. \tag{2.4}$$

Let $\bar{\gamma} : \mathrm{PU}(p) \wedge \mathrm{PU}(p) \to \mathrm{PU}(p)$ be the reduced commutator map. Then there is a map $\tilde{\gamma} : \mathrm{PU}(p) \wedge \mathrm{PU}(p) \to \mathrm{SU}(p)$ such that $\pi \circ \tilde{\gamma} = \gamma$ and $\hat{\gamma} = \tilde{\gamma} \circ (1 \wedge \pi)$. Thus in particular we have

$$\tilde{\gamma}_{p-2} \circ (1 \wedge \cdots \wedge 1 \wedge \pi) = \hat{\gamma}_{p-2}. \tag{2.5}$$

Define a map $\phi : \mathrm{PU}(p) \to \mathrm{SU}(p)$ by $\phi([A]) = A\overline{A}$ for $A \in \mathrm{SU}(p)$. Then we have $\phi^*(x_3) = 2y_3$ and hence by (2.4) and (2.5)

$$\begin{aligned}(\tilde{\gamma}_{p-2} \circ (1 \wedge \cdots \wedge 1 \wedge \pi \circ \phi) \circ \Delta)^*(x_{2p-1}) &= (\hat{\gamma}_{p-2} \circ (1 \wedge \cdots \wedge 1 \wedge \phi) \circ \Delta)^*(x_{2p-1}) \\ &= 2y_2^{p-2}y_3 \neq 0.\end{aligned}$$

This implies that $\tilde{\gamma}_{p-2} \circ (1 \wedge \cdots \wedge 1 \wedge \pi \circ \phi) \circ \Delta$ is essential.

Consider the exact sequence

$$[\mathrm{PU}(p), \mathbf{Z}/p] \to [\mathrm{PU}(p), \mathrm{SU}(p)] \xrightarrow{\pi_*} \mathcal{H}(\mathrm{PU}(n))$$

induced from the covering $\mathbf{Z}/p \to \mathrm{SU}(p) \xrightarrow{\pi} \mathrm{PU}(p)$. Then for $[\mathrm{PU}(p), \mathbf{Z}/p] = *$ we obtain π_* is injective and thus $\pi \circ \bar{\gamma}_{p-2} \circ (1 \wedge \cdots \wedge 1 \wedge \pi \circ \phi) \circ \Delta$ is essential. This is equivalent to that the commutator $\underbrace{[1, [1 \cdots [1,}_{p-2} \pi \circ \phi] \cdots]]$ in $\mathcal{H}(\mathrm{PU}(p))$ is nontrivial and therefore the proof of Theorem 1.1 is completed. □

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